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Note

Relative completeness with respect to two unary functions

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Abstract

One of the most important results is the fact that the number of clones is a continuum for $k \geqslant 3$, while the corresponding set for $k = 2$ is countable. This shows a sharp difference when we go from the binary to the ternary case. This paper discusses the relative completeness with respect to the clone generated by two unary functions and show the sharp difference when we go from four-valued logic to k-valued logic for $k > 4$, as well. The number of maximal clones over a finite set is finite and increases when k increases. However, there are two relative maximal clones if $k = 3, 4$ and there is one relative maximal clone if $k > 4$.

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1. Notation and preliminaries

Denote by $\mathbf{N}$ the set $\{1, 2, \ldots\}$ of positive integers. For $k, n \in \mathbf{N}$, $E_k = \{0, 1, \ldots, k-1\}$, denote by $P_k^{(n)}$ the set of all maps $E_k^n \to E_k$, and $P_k = \bigcup_{n \in \mathrm{N}} P_k^{(n)}$. We say that f is an *i-th projection of arity n* $(1 \leqslant i \leqslant n)$ if $f \in P_k^{(n)}$ and f satisfies the identity $f(x_1, \ldots, x_n) \approx x_i$. We say that $f \in P_k^{(n)}$ is *essential* if it depends on at least two variables and it takes all values from E_k. Let π_i^n denote the i-th projection of arity n, and let Π_k denote the set of all the projections over E_k. For $n, m \geqslant 1, f \in P_k^{(n)}$ and $g_1, \ldots, g_n \in P_k^{(m)}$, *the superposition of* f and $g_1, \ldots, g_n$, denoted by $f(g_1, \ldots, g_n)$, is defined by $f(g_1, \ldots, g_n)(a_1, \ldots, a_m) = f(g_1(a_1, \ldots, a_m), \ldots, g_n(a_1, \ldots, a_m))$ for all $(a_1, \ldots, a_m) \in E_k^m$. A set $F \subseteq P_k$ is *a clone of operations on* E_k (or *clone* for short) if $\Pi_k \subseteq F$ and F is closed with respect to superposition. For $F \subseteq P_k$, $\langle F \rangle_{\mathrm{CL}}$ stands

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for the clone generated by F. We say that clone F is *maximal* if there is no clone G such that $F \subset G \subset P_k$. $F \subseteq P_k$ is *complete* if $\langle F \rangle_{\mathrm{CL}} = P_k$.

Let $\varrho \subseteq E_k^h$ be an h-ary relation and $f \in P_k^{(n)}$. We say that f *preserves* ϱ if for all h-tuples $(a_{11},\dots,a_{1h}),\dots,(a_{n1},\dots,a_{nh})$ from ϱ we have $(f(a_{11},\dots,a_{n1}),\dots, f(a_{1h},\dots,a_{nh})) \in \varrho$. $\mathrm{Pol}\,\varrho$ is the set of all $f \in P_k$ which preserve ϱ. For $F \subseteq P_k$, $\mathrm{Inv}\,F$ denotes the set of all the relations preserved by each $f \in F$.

It is interesting to consider the following problem: What are the maximal clones on a finite universe not containing a given clone C; or, equivalently, what operations to add to C to make it complete(or primal).

The following concept of relative completeness was introduced in [3].

Let C be a clone on E_k and $F \subseteq P_k$. F is *complete relative* to C (or C-*complete*) if $\langle F \cup C \rangle_{\mathrm{CL}} = P_k$.

The following theorem gives a necessary and sufficient condition for F to be C-complete. It is analogous to the Post completeness criterion.

Theorem 1.1 (Tošic et al. [3]). *Let C be a clone on E_k. $F \subseteq P_k$ is complete relative to C if and only if $F \setminus M \neq \emptyset$ for every maximal clone M containing C.*

Therefore, the problem of determining whether a set F is complete relative to C, reduces to determining all the maximal clones that contain C.

This paper heavily depends upon the famous Rosenberg characterization of maximal clones. The following special sets of relations are considered:

R_1— the set of all bounded partial orders on E_k;
R_2— the set of selfdual relations, i.e. relations of the form $\{(x, s(x)) : x \in E_k\}$, where s is a fixed point free permutation of prime order (i.e. $s^p = \mathrm{id}$ for some prime p);
R_3— the set of affine relations, i.e. relations of the form $\{(a,b,c,d) \in E_k^4 : a*b = c*d\}$, where $(E_k, *)$ is a p-elementary Abelian group (p prime);
R_4— the set of all nontrivial equivalence relations on E_k;
R_5— the set of all central relations on E_k;
R_6— the set of all h-regular relations on E_k $(h \geqslant 3)$.

Theorem 1.2 (Rosenberg [2]). *A clone M is maximal iff there is a $\varrho \in R_1 \cup \dots \cup R_6$ such that $M = \mathrm{Pol}\,\varrho$.*

2. Relative completeness

We consider the following operations ([1], Theorem 9, p. 54), on E_k:

$$g(x) = \begin{cases} x, & 0 \leqslant x \leqslant k-3 \\ k-1, & x = k-2 \\ k-2, & x = k-1. \end{cases} \quad ; f(x) = x - 1 (\mathrm{mod}\, k)$$

and the clone generated by them: $C = \langle \{g, f\} \rangle_{\mathrm{CL}}$.

Lemma 2.1. *f preserves no $\varrho \in R_1$.*

Proof. Suppose f preserves $\leqslant_\varrho \in R_1$. Then $\underbrace{f(f(\dots f(}_{k}x)\dots)) = x$ and $x \leqslant_\varrho y \Rightarrow f(x) \leqslant_\varrho f(y)$. Moreover, $x <_\varrho y \Rightarrow f(x) <_\varrho f(y)$. Let a be the least element of $\leqslant_\varrho$. Then $a <_\varrho f(a)$, so we have the following chain of implications:

$$a <_\varrho f(a) \Rightarrow f(a) <_\varrho f(f(a)) \Rightarrow \dots \Rightarrow \underbrace{f(f(\dots f(}_{k-1}a)\dots)) <_\varrho \underbrace{f(f(\dots f(}_{k}a)\dots))$$

i.e.

$$\underbrace{f(f(\dots f(}_{k-1}a)\dots)) <_\varrho a$$

gives a contradiction. □

Lemma 2.2. *For each $\varrho \in R_2$ there is $h \in C$ such that h does not preserve ϱ.*

Proof. Pick $\varrho \in R_2$ such that $\varrho = \{(x, s(x)) \ : \ x \in E_k\}$, where p is a prime number and $s^p = id_{E_k}$. Suppose that f and g preserve ϱ. The following two cases leads to contradiction:

- $(k-1, k-2) \in \varrho$: Then $(g(k-1), g(k-2)) = (k-2, k-1) \in \varrho$, because g preserves ϱ. Therefore, $p = 2$. However, since f preserves ϱ it follows that $(f(k-2), f(k-1)) = (k-3, k-2) \in \varrho$.
- $(k-1, k-2) \notin \varrho$: Then, there exists $a \in E_k \setminus \{k-2\}$ such that $(k-1, a) \in \varrho$. Since g preserves ϱ then $(g(k-1), g(a)) = (k-2, a) \in \varrho$. □

Lemma 2.3. (a) *If $k > 4$ then g preserves no $\varrho \in R_3$.*
(b) *If $k \in \{3, 4\}$ then $\{f, g\}$ preserve $\varrho \in R_3$.*

Proof.
(a) Let ϱ be an affine relation with respect to the Abelian group $(E_k, *, e)$ and suppose g preserves ϱ. Each of the following cases leads to a contradiction:
(1) $e = k-2$: Choose $b, c, d \in \{1, \dots, k-3\}$ such that $b = c * d$. Then $(b, k-2, c, d) \in \varrho$. Since g preserves ϱ, we have $(g(b), g(k-2), g(c), g(d)) = (b, k-1, c, d) \in \varrho$. Now, $b * (k-2) = c * d$ and $b * (k-1) = c * d$ implies $k-1 = k-2$.
(2) $e = k-1$: Choose $b, c, d \in \{1, \dots, k-3\}$ such that $b = c * d$. Then $(b, k-1, c, d) \in \varrho$. Since g preserves ϱ, we have $(g(b), g(k-1), g(c), g(d)) = (b, k-2, c, d) \in \varrho$. Now, $b * (k-1) = c * d$ and $b * (k-2) = c * d$ implies $k-1 = k-2$.
(3) $e \in E_k \setminus \{k-1, k-2\}$: There are $c, d \in \{1, \dots, k-3\}$ such that $k-2 = c * d$. This implies $(c, d, e, k-2) \in \varrho$. Since g preserves ϱ, we have $(g(c), g(d), g(e), g(k-2)) = (c, d, e, k-1) \in \varrho$, i.e., $c * d = e * (k-1)$ and $c * d = e * (k-2)$ implies $k-1 = k-2$.

(b) If $k=3$, then R_3 contains only one maximal set: $Pol\,\varrho = Pol(\{(a,b,c)^{\mathrm{T}} \in E_3^3 | c = 2(a+b)\})$.
Operations from C are permutations on E_3 and obviously they preserve ϱ.

For $k=4$, the only maximal set contained in R_3 is of the form $Pol\,\varrho = Pol(\{(a,b,c,d) \in E_4^4 | a*b = c*d\})$, where $(E_4,*,e)$ is a 2-elementary Abelian group. It can be shown in straightforward way that both f and g preserve ϱ. □

Lemma 2.4. *For each $\varrho \in R_4$ there is $h \in \{f, g\}$ such that h does not preserve ϱ.*

Proof. Choose any $\varrho \in R_4$ and suppose that f and g preserve ϱ. The following four cases are possible:

(1) $card((k-2)/\varrho) > 1 \wedge (k-1) \notin (k-2)/\varrho$: There exists $a \in (k-2)/\varrho$ such that $a \neq k-2$. $(a,k-2) \in \varrho$ implies $(g(a), g(k-2)) = (a, k-1) \in \varrho$, i.e. $(k-1) \in (k-2)/\varrho$.

(2) $card((k-2)/\varrho) > 1 \wedge k-1 \in (k-2)/\varrho$: Since ϱ is a non-trivial equivalence relation, there is $l = \min\{j | j \in (k-1)/\varrho\}$. Thus, $(l-1) \notin (k-1)/\varrho$. From $(k-1,l) \in \varrho$ it follows that $(f(k-1), f(l)) = (k-2, l-1) \notin \varrho$, i.e. $l-1 \in (k-1)/\varrho$.

(3) $card((k-2)/\varrho) = 1 \wedge card((k-1)/\varrho) > 1$: Analogously, for $a \in (k-1)/\varrho, a \neq k-1$, we have that $(a, k-1) \in \varrho$ and $(g(a), g(k-1)) = (a, k-2) \in \varrho$.

(4) $card((k-2)/\varrho) = 1 \wedge card((k-1)/\varrho) = 1$: Since ϱ is a non-trivial equivalence relation, there is $l = \min\{j | card(j/\varrho) > 1\}$. Thus, $j < k-2, card((l-1)/\varrho) = 1$, $card(l/\varrho) > 1$ and there exists $a \in l/\varrho, a \neq l$. From $(a,l) \in \varrho$ it follows that $(f(a), f(l)) = (a-1, l-1) \in \varrho$.
So, in each case we have a contradiction. □

Lemma 2.5. *f preserves no $\varrho \in R_5$.*

Proof. Suppose f preserves a central relation ϱ and choose an arbitrary $(a_1, \ldots, a_h) \notin \varrho$. Since

$$\underbrace{f(f(\ldots(}_{k}x)\ldots)) = x,$$

we have that $(f(a_1), \ldots, f(a_h)) \notin \varrho$. Let c be a central element of ϱ. It can be easily shown that for every $u, v \in E_k$ there is $s \in \mathbf{N}$, such that

$$\underbrace{f(f(\ldots f(}_{s}u)\ldots)) = v.$$

Consider $\ell \in \mathbf{N}$, such that

$$\underbrace{f(f(\ldots f(}_{\ell}a_1)\ldots)) = c$$

and let

$$b_i = \underbrace{f(f(\ldots(}_{\ell}a_i)\ldots)),\ i = 2 \ldots h.$$

Then, $(a_1,\dots,a_h)\notin\varrho$ implies $(c,b_2,\dots,b_h)\notin\varrho$. However, $(c,b_2,\dots,b_h)\in\varrho$ since c is a central element. Contradiction. □

Lemma 2.6. (a) *For each* $\varrho\in R_6, 2<h<k$ *there is* $h\in\{f,g\}$, *such that* h *does not preserve* ϱ.

(b) *The* k*-ary relation* $\varrho\in R_6$ *is preserved by* $\{f,g\}$.

Proof. (a) Let $\varrho\in R_6$ be a h-regular relation, $2<h<k$, determined by a h-regular family of equivalence relations $T=\{q_1,\dots,q_m\}$, and suppose f and g preserve ϱ. Consider the following cases:

(1) $(\exists i_1\in\{1,\dots,m\})(k-2\notin(k-1)/q_{i_1})$:

(1.1) $card((k-1)/q_{i_1})>1$: Denote classes of q_{i_1} by $C_i, 1\leqslant i\leqslant h$, where $C_1=(k-1)/q_{i_1}$ and $C_h=(k-2)/q_{i_1}$. There exists $a_2\in C_1, a_2\in E_k\setminus\{k-1\}$. Since τ is h-regular family of equivalence relations there exists

$$a_{j+1}\in C_j\cap\bigcap_{l=0}^{\lceil m-1/h-2\rceil-1} a_j/q_{i_{l(h-2)+j}}$$

for each $j\in\{2,\dots,h-1\}, a_{j+1}\in E_k\setminus\{a_2,\dots,a_j,k-2,k-1\}$ (if $l(h-2)+j\notin\{2,\dots,m\}$ then suppose that $a_j/q_{i_{l(h-2)+j}}=\emptyset$).

From the definition of h-regular relation and the previous construction it follows that $(k-1,a_2,\dots,a_h)\in\varrho$. Since g preserves ϱ, $(g(k-1),g(a_2),\dots,g(a_h))=(k-2,a_2,\dots,a_h)\in\varrho$ gives a contradiction.

(1.2) $card((k-1)/q_{i_1})=1$: From the definition of a h-regular family of equivalences it follows that $m=1$. We shall prove that f preserves ϱ iff q_1 is an equivalence relation with blocks

$$\{0,h,\dots,(r-1)h\},\{1,h+1,\dots,(r-1)h+1\},\dots,\{h-1,2h-1,\dots,rh-1\},$$

where $r=hk$.

As we have seen in the proof above,

$$(a_1,a_2,\dots,a_h)\notin\varrho\Rightarrow(f(a_1),f(a_2),\dots,f(a_h))\notin\varrho.$$

We can prove that if $x,y\in E_k$ belong to two different classes of q_1, then $f(x)$ and $f(y)$ also belong to different classes of q_1. Suppose that x and y belong to different classes of q_1 and choose $a_3,\dots,a_h\in E_k$ such that $(x,y,a_3,\dots,a_h)\notin\varrho$. Then $(f(x),f(y),f(a_3),\dots,f(a_h))\notin\varrho$, thus proving that $f(x)$ and $f(y)$ belong to different classes of q_1. It is obvious now that if x and y are in the same class C_1, then both $f(x)$ and $f(y)$ belong to some other class $C_2(\neq C_1)$. This implies that the restriction of f to C_1 is a one-to-one mapping into C_2. Furthermore, restriction of

$$\underbrace{f(f(\dots f(}_{k-1}x)\dots))$$

to C_2 is one-to-one mapping into C_1. So, $card\,C_1=card\,C_2$. Going on in this way, we can show that $card\,C_1=card\,C_2=\dots=card\,C_h=r$, where $rh=k$. For every $h|k$ and

$h \notin \{1, 2\}$ there exists only one h-regular family $\tau = \{q_1\}$, i.e., only one relation $\varrho \in R_6$ preserved by f.

Since $card((k-1)/q_{i_1}) = 1$, it follows that f preserves ϱ iff q_1 is an equivalence relation with blocks $\{0\}, \{1\}, \ldots, \{k-1\}$, i.e. iff $h = k$.

(2) $(\forall i \in \{1, \ldots, m\})(k-2) \in (k-1)/q_{i_1}$:

Since $h > 3$, there exists $j \in E_k$ such that $j \notin (j+1)/q_1$. We can choose $a_3, \ldots, a_h \in E_k$ so that $(j, j+1, a_3, \ldots, a_h) \notin \varrho$. Now,

$$(\underbrace{f(f(\ldots f(}_{j+2}j)\ldots)), \underbrace{f(f(\ldots f(}_{j+2}j+1)\ldots)), \underbrace{f(f(\ldots f(}_{j+2}a_3)\ldots)), \ldots, \underbrace{f(f(\ldots f(}_{j+2}a_h)\ldots)))$$
$$= (k-2, k-1, b_3, \ldots, b_h) \notin \varrho, \text{ where } b_i = \underbrace{f(f(\ldots f(}_{j+2}a_i)\ldots)), i \in \{3, \ldots, h\},$$

which gives a contradiction.

(b) Since $Pol\, \varrho$, where $\varrho = E_k^k - P_{01\ldots(k-1)}$, is Slupecki clone, it contains all essential unary functions. $\square$

Theorem 2.1. (a) *If $k > 4$ then there is exactly* 1 *relative maximal clone with respect to C.*

(b) *If $k \in \{3, 4\}$ then there are exactly* 2 *relative maximal clones with respect to C.*

Corollary 2.1. *If $k > 4$ then the set $F \subseteq P_k$ is complete relative to C iff it contains an essential function.*

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